

Crushing candies on the line

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1 The candy crush game

In this paper we investigate stability properties of a probabilistic cellular automaton. The model is based on the candy crush game (see e.g. [1],[2]). The idea of the game is that each point (these are the candies) in a rectangular grid has a certain color. All candies making a horizontal or vertical monochromatic chain disappear simultaneously and candies fall from the top to fill in the gaps. If the resulting configuration again contains such chains, a sequence of reactions occurs and the player is awarded bonus points. This game inspired us to study a somewhat simplified model, where one of the main questions is under what conditions infinite sequences of reactions can occur.

2 Model based on the game

Our model is defined on \mathbb{Z}^d with the Euclidean metric. The points in \mathbb{Z}^d are called sites. We choose n different colors and define the color set $C = \{c_1, \dots, c_n\}$. Furthermore, we choose a *stability constant* $\kappa \in \mathbb{N}, \kappa \geq 2$. A *coloring* or *configuration* is a map $\eta : \mathbb{Z}^d \rightarrow C$. Given the coloring, we will define the notion of stability. Sites $x_1, \dots, x_m \in \mathbb{Z}^d$ satisfying

$$x_i - x_{i-1} = x_2 - x_1, \quad \|x_i - x_{i-1}\| = 1 \quad \text{and} \quad \eta(x_i) = \eta(x_1)$$

for $i = 2, \dots, m$ are said to make a *monochromatic chain* of length m . Sites that are in a monochromatic chain of length greater than or equal to κ are called *unstable*. All other sites are stable. Given a configuration η we define a stability function $\sigma_\eta : \mathbb{Z}^d \rightarrow \{0, 1\}$ by

$$\sigma_\eta(x) = \mathbb{1}_{\{x \text{ is stable in } \eta\}}.$$

A coloring η is stable if and only if $\sigma_\eta(x) = 1$ for all $x \in \mathbb{Z}^d$. A stable configuration in which the color of each site only depends on the parity of the sum of its coordinates will be referred to as a chessboard coloring. The set of all configurations will be denoted by Λ .

In our model, the dynamics are simpler than in the game. We will use a time variable t , taking values in \mathbb{N} . The configuration at time t will be denoted by η_t , so η_0 is the initial configuration. All sites that are unstable at time t (that is, sites x for which $\sigma_{\eta_t}(x) = 0$) will be recolored simultaneously and independently according to some probability distribution p on C to construct η_{t+1} . For $\eta \in \Lambda$, we will denote the random configuration that results from recoloring the unstable sites by $R(\eta)$, so $\eta_{t+1} = R(\eta_t)$.

3 The one-dimensional case with two colors

Let $d = 1$ and $\kappa = 3$. Furthermore, let the color set be coded by $C = \{0, 1\}$ and choose equal recoloring probabilities, so $p = (\frac{1}{2}, \frac{1}{2})$. Then we have the following result:

Theorem 1 *Choose an unstable initial configuration η_0 . Suppose there exists $M \in \mathbb{N}$ such that $\sigma_{\eta_0}(x) = 1$ for all $|x| \geq M$. Then*

$$\mathbb{P}_p(\exists t_0 : \eta_t \text{ is stable for all } t \geq t_0) = 1.$$

Proof. First define the number of unstable sites at time t :

$$I_t = |\{x : \sigma_{\eta_t}(x) = 0\}|, \quad t \geq 0.$$

Note that I_t is bounded by $2M+4t+1$ for all t . We will show that $\lim_{t \rightarrow \infty} I_t = 0$ a.s. We define an upper bounds for the probability that an unstable site is instable again after k time steps as follows:

$$\begin{aligned} p_k^I &= \sup_{\eta \in \Lambda} \mathbb{P}(\sigma_{R^k(\eta)}(x) = 0 \mid \sigma_{\eta}(x) = 0), \\ p_k^{III} &= \sup_{\eta \in \Lambda} \mathbb{P}(\sigma_{R^k(\eta)}(x) = 0 \mid \sigma_{\eta}(x-1) = \sigma_{\eta}(x) = \sigma_{\eta}(x+1) = 0). \end{aligned}$$

By translation invariance, these upper bound do not depend on x . Note that $p_k^{III} \leq p_k^I$. In a similar way we define upper bounds for the probability that a stable site is instable after k recolorings of the configuration. Since this probability highly depends on the distance to instable regions, we condition here on stability of a neighborhood of x :

$$p_k^S(n, m) = \sup_{\eta \in \Lambda} \mathbb{P}(\sigma_{R^k(\eta)}(x) = 0 \mid \prod_{i=-n}^m \sigma_{\eta}(x+i) = 1, \sum_{i=-n-1}^{m+1} \sigma_{\eta}(x+i) = n+m+1),$$

where n, m are non-negative and allowed to take the value ∞ . First we remark that these probabilities are symmetric: $p_k^S(n, m) = p_k^S(m, n)$. Since a stable site can not get instable if a large enough neighborhood is stable, the probabilities $p_k^S(n, m)$ satisfy the following property:

$$p_k^S(n, m) = 0 \quad \text{if} \quad n, m \geq 2k. \tag{1}$$

Furthermore, if there are at least $2k$ stable sites at one side, $p_k^S(n, m)$ does not depend on the exact number of them:

$$\begin{aligned} p_k^S(n_1, m) &= p_k^S(n_2, m) = p_k^S(\infty, m) & \text{if } n_1, n_2 \geq 2k, \\ p_k^S(n, m_1) &= p_k^S(n, m_2) = p_k^S(n, \infty) & \text{if } m_1, m_2 \geq 2k. \end{aligned} \quad (2)$$

Let $\eta \in \Lambda$. A set $\{x, x+1, \dots, x+g\}$ is called a bounded stable region of size g if $\sigma_\eta(x) = \dots = \sigma_\eta(x+g) = 1$ and $\sigma_\eta(x-1) = \sigma_\eta(x+g+1) = 0$. Suppose G is a bounded stable region of size g , then the expected number of sites in G that get instable in k steps is bounded from above by

$$\sum_{i=1}^g p_k^S(i-1, g-i). \quad (3)$$

This sum is the same for all $g > 4k$, since in that case

$$\begin{aligned} \sum_{i=1}^g p_k^S(i-1, g-i) &= \sum_{i=1}^{2k} p_k^S(i-1, \infty) + \sum_{i=g-2k+1}^g p_k^S(\infty, g-i) \\ &= \sum_{i=1}^{2k} p_k^S(i-1, 4k-i) + \sum_{i=2k+1}^{4k} p_k^S(i-1, 4k-i) \\ &= \sum_{i=1}^{4k} p_k^S(i-1, 4k-i), \end{aligned}$$

by the properties (1) and (2). If η is such that $\sigma_\eta(x) = 1$ and $\sigma_\eta(y) = 0$ for all $y < x$, then the set $\{y \in \mathbb{Z} : y < x\}$ is called a left-unbounded stable region. A right-unbounded stable region is defined similarly. For an unbounded stable region, the expected number of sites that gets instable is at most

$$\sum_{i=1}^{\infty} p_k^S(i-1, \infty) = \sum_{i=1}^{2k} p_k^S(i-1, \infty) = \sum_{i=2k+1}^{4k} p_k^S(4k-i, \infty).$$

where we used (1) for the second equality. Exploiting (2), we arrive at

$$\begin{aligned} \sum_{i=1}^{\infty} p_k^S(i-1, \infty) &= \frac{1}{2} \sum_{i=1}^{2k} p_k^S(i-1, 4k-i) + \frac{1}{2} \sum_{i=2k+1}^{4k} p_k^S(4k-i, i-1), \\ &= \frac{1}{2} \sum_{i=1}^{4k} p_k^S(i-1, 4k-i). \end{aligned}$$

Since an instable region consists of at least 3 sites, at least a third of the unstable sites has two unstable neighbors. For the same reason, the number

of bounded stable regions in η_t is at most $I_t/3 - 1$. Furthermore, since I_t is finite, there are two unbounded stable regions in η_t . Therefore

$$\begin{aligned}\mathbb{E}[I_{t+k}|I_t] &\leq \frac{1}{3}p_k^{III}I_t + \frac{2}{3}p_k^I I_t + \left(\frac{I_t}{3} - 1\right) \max_{1 \leq g \leq 4k} \sum_{i=1}^g p_k^S(i-1, g-i) + \sum_{i=1}^{4k} p_k^S(i-1, 4k-i) \\ &\leq \left(\frac{1}{3}p_k^{III} + \frac{2}{3}p_k^I + \frac{1}{3} \max_{1 \leq g \leq 4k} \sum_{i=1}^g p_k^S(i-1, g-i) \right) I_t.\end{aligned}\tag{4}$$

Our next goal is to show that there exists k for which the constant in front of I_t is smaller than 1. In order to do this, we compute p_k^I and $p_k^S(n, m)$ for $0 \leq n, m < 2k$ for some values of k .

First we take $k = 1$. Let $\eta \in \Lambda$. Then $\mathbb{P}(\sigma_{R^1(\eta)}(0) = 0)$ only depends on $\eta(x)$ and $\sigma_\eta(x)$ for $-2 \leq x \leq 2$. So to find the probabilities p_k^I and $p_k^S(n, m)$, we can just check all possibilities. In the table below we listed them for the case $\sigma_\eta(0) = 0$, without loss of generality assuming that $\eta(0) = 0$ and omitting (symmetrically) equivalent cases:

$(\eta(x))_{x=-2}^{x=2}$	$(\sigma_\eta(x))_{x=-2}^{x=2}$	$\mathbb{P}(\sigma_{R^1(\eta)}(0) = 0)$
00000	00000	1/2
00001	00001	1/2
00010	00010	1/2
00010	00011	3/8
00011	00011	5/8
10001	10001	1/2

It follows that $p_1^I = 5/8$ and $p_1^{III} = 1/2$. As a second example, we compute $p_1^S(1, 2)$. So here we maximize $\mathbb{P}(\sigma_{R^1(\eta)}(0) = 0)$ over configurations η for which $\sigma_\eta(-2) = 0$ and $\sigma_\eta(x) = 1$ if $-1 \leq x \leq 2$. Again we assume that $\eta(0) = 0$. Then the following cases are possible:

$(\eta(x))_{x=-2}^{x=2}$	$\mathbb{P}(\sigma_{R^1(\eta)}(0) = 0)$
01001	0
01010	0
01011	0
10010	1/2
10011	1/2

Therefore, $p_1^S(1, 2) = 1/2$. For other values of n and m a similar calculation leads to the following values of $p_1^S(n, m)$:

	$m = 0$	$m = 1$	$m = 2$
$n = 0$	1/2	3/4	1/2
$n = 1$	3/4	1/2	1/2
$n = 2$	1/2	1/2	0

Using (2) the maximal value of the sum in (4) turns out to be equal to 2 (for $g = 4$), whence

$$\mathbb{E}[I_{t+1}|I_t] \leq \left(\frac{1}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{5}{8} + \frac{1}{3} \cdot 2 \right) I_t = \frac{5}{4} I_t.$$

For other values of k , we can proceed analogously. The key observation is that $\mathbb{P}(\sigma_{R^k(\eta)}(0) = 0)$ is completely determined by $\eta(x)$ and $\sigma_\eta(x)$ for $-2k \leq x \leq 2k$. For $k = 4$, we find the desired inequality $\mathbb{E}[I_{t+k}|I_t] \leq cI_t$ with $c < 1$. Below is a summary of the computational results.

$$\begin{aligned} \mathbb{E}[I_{t+2}|I_t] &\leq \left(\frac{1}{3} \cdot \frac{29}{64} + \frac{2}{3} \cdot \frac{61}{128} + \frac{1}{3} \cdot \frac{19}{8} \right) I_t = \frac{121}{96} I_t, \\ \mathbb{E}[I_{t+3}|I_t] &\leq \left(\frac{1}{3} \cdot \frac{5037}{16384} + \frac{2}{3} \cdot \frac{2687}{8192} + \frac{1}{3} \cdot \frac{2495}{1024} \right) I_t = \frac{55705}{49152} I_t, \\ \mathbb{E}[I_{t+4}|I_t] &\leq \left(\frac{1}{3} \cdot \frac{15371121}{67108864} + \frac{2}{3} \cdot \frac{518955}{2097152} + \frac{1}{3} \cdot \frac{2371247}{1048576} \right) I_t = \frac{200344049}{201326592} I_t. \end{aligned}$$

The values of $p_4^S(n, m)$ can be written as fractions. Their numerators are given in the table below. Entries in the same column have the same denominator, which is written in the second line of the table. All denominators are powers of 2. For example $p_4^S(5, 6) = p_4^S(6, 5) = \frac{358}{2048} (= \frac{179}{1024})$.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
denom.	536870912 ($=2^{29}$)	67108864 ($=2^{26}$)	67108864 ($=2^{26}$)	4194304 ($=2^{22}$)	131072 ($=2^{17}$)	2048 ($=2^{11}$)	512 ($=2^9$)	16 ($=2^4$)	1 ($=2^0$)
$m=0$	109921252								
$m=1$	125921345	14557783							
$m=2$	116330096	17164756	15205719						
$m=3$	131398816	15993296	18114414	926907					
$m=4$	118712000	17449960	14234240	1036746	21623				
$m=5$	128603456	15641520	16272448	860952	27214	307			
$m=6$	112440704	16537568	12895872	948720	17728	358	47		
$m=7$	119681024	14745472	14295552	773184	21120	256	62	1	
$m=8$	105200384	14745472	11496192	773184	14336	256	32	1	0

Now we have the inequality

$$\mathbb{E}[I_{t+4}|I_t] \leq cI_t, \quad \text{with } c = \frac{200344049}{201326592} < 1.$$

Taking expectations,

$$\mathbb{E}[I_{t+4}] = \mathbb{E}[\mathbb{E}[I_{t+4}|I_t]] \leq c\mathbb{E}[I_t].$$

Therefore, for all $t \in \mathbb{N}$, we obtain

$$\mathbb{E}[I_{4t}] \leq c^t \mathbb{E}[I_0] \leq c^t (2M + 1).$$

Since I_{4t} is positive integer-valued and the events $\{I_{4t} \geq 1\}$ are decreasing, we get

$$\begin{aligned}\mathbb{P}(\exists t : I_{4t} = 0) &= \mathbb{P}\left(\bigcup_{t=0}^{\infty} \{I_{4t} = 0\}\right) = 1 - \mathbb{P}\left(\bigcap_{t=0}^{\infty} \{I_{4t} \geq 1\}\right) \\ &= 1 - \lim_{t \rightarrow \infty} \mathbb{P}(\{I_{4t} \geq 1\}) \geq 1 - \lim_{t \rightarrow \infty} \mathbb{E}[I_{4t}] \\ &\geq 1 - \lim_{t \rightarrow \infty} c^t(2M + 1) = 1,\end{aligned}$$

where we used Markov's inequality. If $I_{4t_0} = 0$, then η_t is stable for all $t \geq 4t_0$. Therefore

$$\mathbb{P}(\exists t_0 : \eta_t \text{ is stable for all } t \geq t_0) = 1.$$

□

References

- [1] <http://www.games.com/play/king/candy-crush>
- [2] http://en.wikipedia.org/wiki/Candy_Crush_Saga